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Research paper

An equilibrium analysis of the Arad-Rubinstein game☆

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ABSTRACT

Colonel Blotto games with discrete strategy spaces nicely illustrate the intricate nature of multidimensional strategic reasoning. This paper studies the equilibrium set of such games where, in line with prior experimental work, the tie-breaking rule is allowed to be flexible. We begin by pointing out that equilibrium constructions known from the literature extend to our class of games. However, we also note that, irrespective of the tie-breaking rule, the equilibrium set is excessively large. Specifically, any pure strategy that allocates at most twice the fair share to each battlefield is used with positive probability in some equilibrium. Furthermore, refinements based on the elimination of weakly dominated strategies prove ineffective.

1. Preliminaries

1.1. Introduction

In a Colonel Blotto game, as envisaged by Borel (1921), two adversaries are tasked with allocating their budgets of a resource secretly over a given set of battlefields, aiming to conquer as many battlefields as possible. In each battlefield, victory is awarded to the adversary allocating a higher amount of the resource, where a tie-breaking rule is invoked when both parties assign an equal amount. Applications of Colonel Blotto games extend beyond military conflicts to areas such as strategic marketing, electoral competition, innovation contests, and network security. Key contributions in the literature, including Roberson (2006) for continuous and Hart (2008) for discrete strategy spaces, assumed that the Colonel Blotto game is *constant-sum at ties*, i.e., that any tied battlefield is ultimately conquered by one contestant or another, but never lost. This characteristic of the standard model ensures that the Colonel Blotto game is a two-person constant-sum game, significantly simplifying the equilibrium analysis. Notably, however, the assumption is not satisfied in experimental studies such as Arad and Rubinstein's (2012) investigation of multidimensional strategic reasoning. In their case, tied battlefields count as lost for both parties, rendering the standard equilibrium characterization invalid.[1]

The present study investigates a class of Colonel Blotto games characterized by discrete strategy spaces and flexible tie-breaking rules. Our model nests both the constant-sum version of the game and its variant where tied battlefields generate no value. We

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[1] See Arad and Rubinstein (2012, p. 584) for their statement: "We are not aware of any analysis of the Nash equilibria of our version of the game".

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present an approach that allows constructing Nash equilibria in this wider class of games. In particular, we find an equilibrium in the example considered by Arad and Rubinstein (2012). However, a caveat emerges. Specifically, our analysis reveals that the model admits an excessively large set of equilibria regardless of the tie-breaking rule. Furthermore, attempts to narrow the equilibrium set down by applying the concept of weak dominance prove ineffective.

1.2. Contribution

The present study contributes to the literature on finite Blotto games in two main ways. Firstly, we construct Nash equilibria in the model with flexible tie-breaking. For the constant-sum game, Hart (2008) demonstrated that marginal distributions on individual battlefields can be chosen to be essentially uniform. Consequently, assuming divisibility, players may partition the set of battlefields into pairs, evenly distribute their resources across those pairs, and randomize uniformly within each pair. We observe that the precise form of the tie-breaking rule is less crucial for the equilibrium property, as the probability of a tie remains constant across bid levels used in equilibrium. We also point out that the loss of payoffs due to ties, when compared to the constant-sum version, is insufficient to make overbidding an attractive strategy. Based on these observations, we establish equilibria in the model with flexible tie-breaking, specifically in the Arad–Rubinstein game. In doing so, we identify conditions under which the equilibrium initially identified by Hart (2008) persists in games with modified tie-breaking.

Our second main contribution is the observation that, irrespective of the tie-breaking rule, the equilibrium set of games in the considered class is excessively large. More specifically, we show that *any* pure strategy that does not put too many resources on any individual battlefield is part of some equilibrium strategy. The idea of identifying pure strategies that are part of some mixed equilibrium arises already in Tukey (1949), who termed those strategies “good”. We find that any pure strategy that allocates at most twice the fair share of the budget to each battlefield has this property.[2] Moreover, none of these strategies can be eliminated by equilibrium refinements based on the elimination of weakly dominated strategies.[3]

1.3. Related literature

In a seminal paper on Colonel Blotto games, Borel (1921) considered both continuous and finite strategy spaces. For the model with continuous strategy spaces, Roberson (2006) characterized optimal marginal distributions and unique equilibrium payoffs. He also noted that in the continuous model, tie-breaking rules often do not matter (but may need modification to ensure existence).

For the model with discrete strategy spaces, Hart (2008) constructed equilibria not only in all cases with homogeneous endowments, but also for special cases with heterogeneous endowments. He used two auxiliary models. In the *Colonel Lotto game*, players can be thought of as being restricted to mixed strategies that are invariant under arbitrary permutations of the set of battlefields. In the *General Lotto game*, the budget constraint needs to be satisfied in expectation only. Solutions of the General Lotto game turn into solutions of the corresponding Colonel Lotto and Colonel Blotto games provided that those solutions are feasible, i.e., marginals can be derived from a joint distribution that satisfies the budget constraint with equality. Building on these concepts, Dziubiński (2013, 2017) characterized the set of optimal marginal distributions in the General Lotto game and, provided that the number of resources is divisible by the number of battlefields, also the set of optimal marginal distributions of Colonel Blotto games with discrete strategy spaces. Despite this progress, the general characterization of the equilibrium set in Colonel Blotto games with discrete strategy spaces has remained elusive. More recently, Liang et al. (2023) and Aspect and Ewerhart (2022) characterized equilibria in discrete Colonel Blotto games with two battlefields. As for as we know, however, prior work has not attempted to characterize the equilibrium in a discrete Colonel Blotto game with a modified tie-breaking rule.[4]

We refer to Arad and Rubinstein (2019) and the literature cited therein regarding the procedures of players who think strategically *in categories*. That approach is supported by the experimental literature following Arad and Rubinstein (2012). Experimental tests of Blotto games not mentioned above include Avrahami and Kareev (2009), Kohli et al. (2012), Chowdhury et al. (2013), Avrahami et al. (2014), and Montero et al. (2016), among others.

1.4. Overview of the paper

The remainder of the paper is structured as follows. Section 2 introduces the model. In Section 3, we present an approach for constructing Nash equilibria within the considered class of games. Section 4 concerns the equilibrium set. Section 5 deals with refinements. Section 6 offers further discussion. Section 7 concludes.

[2] Conversely, strategies that concentrate the resource on too few battlefields are never “good”.

[3] The analysis also leads to a number of supplementary observations on the constant-sum model, which we report upon in a separate section before the conclusion.

[4] However, Rapoport and Amaldoss (2000) and Dechenaux et al. (2006) considered an all-pay auction with discrete bids and a modified tie-breaking rule.

2. The model

2.1. Setup and notation

Two players, denoted by A and B, each allocate a total of $N \geq 1$ units of a resource over $K \geq 2$ battlefields.[5] Units of the resource are not divisible. Hence, a *pure strategy* of player i is a vector

$$s^i = \begin{pmatrix} s_1^i \\ \vdots \\ s_K^i \end{pmatrix},$$

such that $s_k^i \in \{0, 1, \ldots, N\}$ for every $k \in \{1, \ldots, K\}$, and

$$\sum_{k=1}^{K} s_k^i = N.$$

The set of strategies for players A and B is identical and denoted by S. As usual, we refer to the opponent of player i by $-i$. For a given pure strategy profile $(s^i, s^{-i}) \in S \times S$, the payoff of player $i \in \{A, B\}$ in the Colonel Blotto game is defined by

$$\pi^i\left(s^i, s^{-i}\right) = \sum_{k=1}^{K} \left(\mathbb{1}_{s_k^i > s_k^{-i}} + \frac{\alpha}{2} \cdot \mathbb{1}_{s_k^i = s_k^{-i}} \right),$$

where $\mathbb{1}_{s_k^i > s_k^{-i}}$ equals one if player i's bid in battlefield k exceeds that of player $-i$, and zero otherwise, $\mathbb{1}_{s_k^i = s_k^{-i}}$ equals one in the case of a tie on battlefield k, and zero otherwise, and α is a parameter. The departure from the standard model is the introduction of flexible tie-breaking, represented by α. We call the two-player game with strategy sets and payoffs defined as above the *Colonel Blotto game* $\mathcal{B}_\alpha \equiv \mathcal{B}_\alpha(N, K)$.[6]

2.2. Examples

Below, we recall two examples of finite Colonel Blotto games that have been considered in the literature.

Example 1 (*Hart, 2008*). In the standard version of the Colonel Blotto game, $\mathcal{B}_1(N, K)$, player i's payoff is defined as

$$\pi^i\left(s^i, s^{-i}\right) = \sum_{k=1}^{K} \left(\mathbb{1}_{s_k^i > s_k^{-i}} + \frac{1}{2} \cdot \mathbb{1}_{s_k^i = s_k^{-i}} \right).$$

The constant-sum setup underlying Example 1 has indeed been prevalent in the literature. Borel (1921, p. 100) considered a particular case where $N = 7$ and $K = 3$, a solution of which has been offered by Hart (2008). The setup in Example 1 has been tested as a symmetric control by Avrahami et al. (2014) for $N \in \{16, 24\}$ and $K = 8$.

Example 2 (*Arad and Rubinstein, 2012*). Two colonels are asked to distribute a total of $N = 120$ units of the resource over a total of $K = 6$ battlefields. The payoff of a player is the number of battlefields that she assigned strictly more resources to than her opponent. Thus, the game is $\mathcal{B}_0(120, 6)$, the common set of strategies is given as $S = \{s^i \in \{0, 1, \ldots, 120\}^6 : \sum_{k=1}^{6} s_k^i = 120\}$, and the payoff of player i is defined by

$$\pi^i\left(s^i, s^{-i}\right) = \sum_{k=1}^{6} \mathbb{1}_{s_k^i > s_k^{-i}}.$$

Abstracting from the fact that Example 1 keeps N and K flexible while Example 2 does not, the main difference between the two examples lies in the tie-breaking rule applied. In contrast to Example 1, the game in Example 2 fails to be constant-sum, as the sum of payoffs of both players depends on probabilities of ties occurring on individual battlefields. This complicates the equilibrium analysis but also makes the game more interesting. Intuitively, setting $\alpha = 0$ provides an additional incentive to outguess the opponent. For instance, in an experiment with $\alpha = 1$, subjects might perceive the pure strategy $(20, 20, 20, 20, 20, 20)'$ as a focal point that leads to a fair division.[7] With $\alpha = 0$, however, payoffs at this focal point are zero for both players, i.e., there is a strong incentive to engage in at least *some* additional reasoning. As this consideration is mentioned by Arad and Rubinstein (2012) early in their work, it might have contributed to their decision to depart from the standard tie-breaking rule.

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[5] In the excluded cases where $N = 0$ or $K = 1$, players have only one strategy. However, note that we allow for $K = 2$, which is not necessarily trivial in our model.

[6] The number of pure strategies in $\mathcal{B}_\alpha(N, K)$ is $|S| = \binom{N+K-1}{K-1}$. In the corresponding Colonel Lotto game, i.e., taking account of symmetries between battlefields, the number of pure strategies is given by the number of partitions of N into exactly K *weakly* positive parts, or equivalently, by $p(N + K, K)$, where $p(N, K)$ denotes the number of partitions of N into exactly K *strictly* positive parts. The recursive relationship

$$p(N, K) = p(N - 1, K - 1) + p(N - K, K)$$

with initial conditions $p(N, 1) = 1$ and $p(N, K) = 0$ if $K > N$ allows computing this number in specific examples (Gupta, 1970).

[7] Here and below, we represent the strategy as a row vector via transposition.

2.3. Assumptions

For expositional reasons, we will initially work under a set of simplifying assumptions. The implications of dropping these assumptions will be discussed in the extensions section. Our first assumption concerns the parity of the number of battlefields.

Assumption 1. K is even, i.e., $K = 2L$ for some integer L.

Assumption 1 simplifies the analysis but can often be dropped at the cost of additional arguments. Experimental papers tend to work under the assumption. Our second assumption concerns the relationship between N and K.

Assumption 2. N is divisible by K.

Thus, in the main part of the analysis, we will assume that the number of resources N is a multiple of K. It follows from Assumption 2 that

$$m = \frac{N}{K}$$

is an integer. For an experimental subject, this means that the uniform allocation that assigns m units of the resource to each of the battlefields is a possibility. E.g., Assumption 2 is violated in Avrahami and Kareev (2009), but it holds in Avrahami et al. (2014). The parameter m will play an important role in the sequel. The same is true for the efficiency parameter α on which we impose the following assumption.

Assumption 3. $\alpha \in [0, 2]$.

Assumption 3 may be considered natural but nevertheless restricts the efficiency parameter in two ways. The restriction $\alpha \geq 0$ says that ties cannot cause inefficiencies beyond the complete loss of the battlefield value. For $\alpha < 0$, players would have a very strong incentive to avoid ties, which may lead to asymmetric equilibria. Conversely, for $\alpha > 2$, ties would be "superefficient", so that pure-strategy equilibria become natural. Such possibilities will be further discussed in the extensions section.

Note that Assumptions 1 through 3 hold in the Arad–Rubinstein game. Indeed, in Example 2, $K = 6$ is even, $m = \frac{120}{6} = 20$ is an integer, and $\alpha = 0 \in [0, 2]$.

2.4. Equilibrium concept

Given a finite non-empty set X, let $\Delta(X)$ denote the set of all probability distributions on X. We are interested in *mixed-strategy Nash equilibria* of the game, i.e., mixed strategy profiles $\sigma = (\sigma^i, \sigma^{-i}) \in \Delta(S) \times \Delta(S)$ such that no player can improve her expected payoff by unilaterally changing her mixed strategy. By a *symmetric Nash equilibrium strategy*, we mean any mixed strategy σ^i such that $\sigma = (\sigma^i, \sigma^i)$ is a mixed-strategy Nash equilibrium.

A crucial property of Colonel Blotto games is that payoffs are functions of the respective marginal distributions at each battlefield. Given a mixed strategy σ^i and a battlefield $k \in \{1, \ldots, K\}$, let σ^i_k denote the marginal distribution of σ^i at battlefield k. Following Hart (2008), we denote the uniform marginal on $\{0, \ldots, 2m\}$ by U^m. A mixed strategy σ^i such that $\sigma^i_k = U^m$ for every battlefield k will be said to induce *uniform marginals*.

3. Equilibrium in the Colonel Blotto game with flexible tie-breaking

In this section, we present a simple approach that allows constructing an equilibrium in the Colonel Blotto game with flexible tie-breaking. As mentioned before, this will lead us to an equilibrium in the Arad–Rubinstein game as well.

3.1. A canonical equilibrium

The following result characterizes one particular Nash equilibrium in $B_\alpha(N, K)$.

Proposition 1. *Impose Assumptions 1 through 3. Then, a symmetric equilibrium strategy of $B_\alpha(N, K)$ is given by uniform randomization over the set of pure strategies*

$$S_0 = \left\{ \begin{pmatrix} 0 \\ 2m \\ \vdots \\ 0 \\ 2m \end{pmatrix}, \begin{pmatrix} 1 \\ 2m-1 \\ \vdots \\ 1 \\ 2m-1 \end{pmatrix}, \ldots, \begin{pmatrix} 2m \\ 0 \\ \vdots \\ 2m \\ 0 \end{pmatrix} \right\}.$$

In the resulting equilibrium, players' expected payoffs amount to $\pi^* = K \cdot \frac{m + \frac{\alpha}{2}}{2m+1}$.

Proof. See Section 3.4. $\square$

As can be seen, players partition the set of battlefields into pairs. This is possible, of course, because the number of battlefields has been assumed even via Assumption 1.[8] To each pair of battlefields, a constant number of $2m$ resources is allocated, which is feasible due to Assumption 2. Moreover, the split among the two battlefields in each pair is uniformly distributed, with perfect correlation across pairs. We note that players' strategies induce uniform marginals. Indeed, for any battlefield k, every number of resources in $\{0, 1, 2, \ldots, 2m\}$ is assigned to battlefield k with the same probability of $\frac{1}{2m+1}$.

3.2. Discussion

Hart (2008) has shown that, under Assumptions 1 and 2, strategies inducing uniform marginals form an equilibrium in the constant-sum game $B_1(N, K)$. The point to note is that this remains the case even with flexible tie-breaking. The reason why the equilibrium property does not break down with more flexible tie-breaking, and this is the main observation that motivated our work on the present paper, is that, given uniform marginals, *the likelihood of getting tied is constant across all bid levels that are used in equilibrium with positive probability*. As a result, the indifference in the standard setup with $\alpha = 1$ is not affected by the modification of the payoff functions at ties.

One might wonder if, with the modified tie-breaking in place, players would not have an incentive to bid higher than $2m$ on some of the battlefields. Such an incentive might arise for $\alpha < 1$, because the tie-breaking is inefficient in that case. However, overbidding cannot raise a player's payoff. The reason is that any additional unit of the resource required to assign more than $2m$ on some battlefield needs to be taken from some other battlefield, where this lowers the probability of winning by $\frac{1}{2m+1}$. Indeed, every bid level in $\{0, 1, \ldots, 2m\}$ is used by the opponent on every battlefield with the same probability of $\frac{1}{2m+1}$. On the other hand, the increase in payoff from bidding $2m + 1$ instead of $2m$ in a battlefield is $\frac{1}{2m+1} \times (1 - \alpha) \leq \frac{1}{2m+1}$. Therefore, given Assumption 3, or more precisely given that $\alpha \geq 0$, the deviation never yields a strictly higher payoff.[9]

3.3. Illustration

We illuminate Proposition 1 with an example.

Corollary 1 (*Equilibrium in the Arad–Rubinstein Game*). *The following strategy is a symmetric Nash equilibrium strategy in $B_0(120, 6)$. Both players individually and independently randomize uniformly over the set*

$$S_0 = \left\{ \begin{pmatrix} 0 \\ 40 \\ 0 \\ 40 \\ 0 \\ 40 \end{pmatrix}, \begin{pmatrix} 1 \\ 39 \\ 1 \\ 39 \\ 1 \\ 39 \end{pmatrix}, \ldots, \begin{pmatrix} 40 \\ 0 \\ 40 \\ 0 \\ 40 \\ 0 \end{pmatrix} \right\}.$$

The equilibrium payoff is $\pi^* = \frac{120}{41} \approx 2.927$.

Proof. Immediate from Proposition 1. □

Even though the equilibrium characterized by Proposition 1 has a canonical structure (e.g., it has uniform marginals, and is symmetric with respect to permutations of the battlefield pairings and within battlefield pairings), it is not the only equilibrium, as will be shown in the next section.

3.4. Proof of Proposition 1

We start with an auxiliary result.

Lemma 1. *Impose Assumptions 2 and 3. Let $\sigma = (\sigma^A, \sigma^B)$ be a mixed strategy profile in $B_\alpha(N, K)$ such that σ^i induces uniform marginals for $i \in \{A, B\}$. Then, σ is a Nash equilibrium.*

Proof. Suppose that both players use the assumed mixed strategies that induce uniform marginals. Then, the probability of a tie on any given battlefield is $\frac{1}{2m+1}$. Therefore, each player i's expected payoff from $\sigma = (\sigma^A, \sigma^B)$ equals

$$\pi^i(\sigma) = \frac{K}{2} + K \cdot \frac{1}{2m+1} \cdot \frac{\alpha - 1}{2} = \frac{K \cdot (2N + \alpha K)}{4N + 2K}.$$

[8] We would like to add, however, that Assumption 1 can be dropped without loss, as follows from Dziubiński (2017, Prop. 2). Then, however, the equilibrium strategies are more involved, as we illustrated with an example in the working paper version (Ewerhart and Kaźmierowski, 2024).

[9] As the discussion shows, the conclusion of Proposition 1 remains technically true for $\alpha > 2$. However, as mentioned before, there is no reason to hide one's strategy for $\alpha \geq 2$, i.e., symmetric pure-strategy equilibria may be more plausible in that case. Cf. Section 6.

Suppose that player $i \in \{A, B\}$ deviates to a pure strategy $s^i \in S$. The mixed strategy σ^{-i} allocates every number of resources in $\{0, 1, 2, \ldots, 2m\}$ on every battlefield k with the same probability of $\frac{1}{2m+1}$. If player i assigns strictly more than $2m$ resources to a battlefield, her payoff from that battlefield is 1. If player i assigns weakly less than $2m$ to battlefield k, then she overwhelms the opponent on that battlefield with probability $\frac{s^i_k}{2m+1}$ and achieves a tie with probability $\frac{1}{2m+1}$. Hence, player i's expected payoff from battlefield k is

$$\pi^i_k\left(s^i, \sigma^{-i}\right) = \min\{1, \frac{s^i_k}{2m+1} + \frac{\alpha}{2} \cdot \frac{1}{2m+1}\}.$$

It follows that

$$\begin{aligned}
\pi^i\left(s^i, \sigma^{-i}\right) &= \sum_{k=1}^{K} \pi^i_k\left(s^i, \sigma^{-i}\right) \\
&\leq \sum_{k=1}^{K} \left(\frac{s^i_k}{2m+1} + \frac{\alpha}{2} \cdot \frac{1}{2m+1}\right) \\
&= \frac{N}{2m+1} + \frac{K \cdot \alpha}{2 \cdot (2m+1)} \\
&= \frac{K \cdot (2N + \alpha K)}{4N + 2K}.
\end{aligned} \tag{1}$$

As both i and s^i were arbitrary, no player can raise her payoff by deviating from $\sigma = (\sigma^A, \sigma^B)$. Hence, σ is a Nash equilibrium, which completes the proof of the lemma. □

Proof for Proposition 1. The mixed strategy σ^i, which is well-defined given Assumption 1, assigns every number of resources $s^i_k \in \{0, 1, \ldots, 2m\}$ to every battlefield k with the same probability $\frac{1}{2m+1}$. Hence, σ^i induces uniform marginals. It therefore follows from Lemma 1 above that σ^i is a symmetric equilibrium strategy. □

4. Understanding the equilibrium set

The equilibrium set turns out to be very large. To make this point, we will study the support of arbitrary randomized equilibrium strategies. We first identify pure strategies that are chosen with positive probability in some mixed-strategy Nash equilibrium (Section 4.1). Then, we outline the proof (Section 4.2). Finally, we turn to strategies that are never "good" (Section 4.3).

4.1. Pure strategies that arise in some equilibrium

The following result gives an indication of the size of the equilibrium set.

Proposition 2. *Impose Assumptions 1 through 3. Then, every pure strategy s^i such that $s^i_k \leq \frac{2N}{K}$ for every battlefield k is used with positive probability in some randomized equilibrium strategy of $B_\alpha(N, K)$.*

Proof. See Section 4.2. □

Thus, every pure strategy that does not allocate an excessive number of resources to an individual battlefield is used in some mixed-strategy Nash equilibrium.

In the Arad–Rubinstein game $B_0(120, 6)$, every pure strategy that assigns at most 40 soldiers to each of the battlefields is part of some mixed strategy Nash equilibrium. This observation illustrates a drawback of the Nash equilibrium concept for the analysis of Colonel Blotto games. The set of equilibrium predictions is simply very large. We will come back to this issue in the next section.

Proposition 2 relates to observations made by Tukey (1949) saying that "there are good strategies in which a given player either (i) sends out no units, (ii) sends out more than half of some kind of unit, or (iii) sends units to more than half of the available sites".[10] Interpreting "good" as appearing in the support of some randomized equilibrium strategy, it is not hard to see that Proposition 2 implies conditions (i) and (iii) under the assumptions of the present paper. An illustration of the possibility of condition (ii) can be found in the working paper version (Ewerhart and Kaźmierowski, 2024, Ex. 3).

4.2. Proof of Proposition 2

To understand why Proposition 2 is true, suppose that both players uniformly randomize over the set of pure strategies

$$S_1 = \left\{ \begin{pmatrix} s_1 \\ 2m - s_1 \\ \vdots \\ s_L \\ 2m - s_L \end{pmatrix} : s_1, \ldots, s_L \in \{0, 1, \ldots, 2m\} \right\}.$$

[10] These general observations hold, in particular, for an asymmetric version of the Colonel Blotto game discussed in McDonald and Tukey (1949).

Clearly, this strategy induces uniform marginals. But this implies, by the discussion following Proposition 1, that both players are actually using an equilibrium strategy.[11]

Let s be any pure strategy such that $s_k \leq 2m$ for all $k \in \{1, \ldots, K\}$. Then, it suffices to replace the two pure strategies

$$\begin{pmatrix} s_1 \\ 2m - s_1 \\ \vdots \\ s_{K-1} \\ 2m - s_{K-1} \end{pmatrix}, \begin{pmatrix} 2m - s_2 \\ s_2 \\ \vdots \\ 2m - s_K \\ s_K \end{pmatrix}$$

in the support of the symmetric equilibrium strategy identified in Proposition 1 by

$$\begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_{K-1} \\ s_K \end{pmatrix}, \begin{pmatrix} 2m - s_2 \\ 2m - s_1 \\ \vdots \\ 2m - s_K \\ 2m - s_{K-1} \end{pmatrix},$$

respectively. The marginals do not change, and hence, we have found an equilibrium strategy in which s is played with positive probability. This concludes the argument. □

4.3. *Strategies that are never "good"*

In analogy to Proposition 2, one may ask what type of pure strategy is *never* used in equilibrium. To address this question, we identify the maximum loss of efficiency that is feasible with modified tie-breaking. Then, we derive conditions on pure strategies that make them render an expected payoff too low to correspond to the maximum efficiency loss. Proceeding along these lines, we show that any strategy that focuses on too few battlefields will never be part of any mixed-strategy Nash equilibrium.

Proposition 3. *Impose Assumptions 1 and 2, and let $\alpha \in [0,1)$. Then, any pure strategy that allocates the resource to strictly less than*

$$K^* = \frac{NK}{2N+K}\left(1 - \frac{\alpha}{2-\alpha}\right)$$

battlefields is never part of any equilibrium of $B_\alpha(N,K)$.[12]

Proof. Consider an arbitrary pure strategy s^i that allocates a positive number of units of the resource to $K_+ \in \{1, \ldots, K\}$ battlefields and zero units to the remaining battlefields. Then, player i's expected payoff of s^i against an arbitrary pure strategy s^{-i} satisfies the relationship

$$\begin{aligned} \pi^i\left(s^i, s^{-i}\right) &\leq K_+ \cdot 1 + (K - K_+) \cdot \frac{\alpha}{2} \\ &= \frac{\alpha}{2} \cdot K + \frac{2-\alpha}{2} \cdot K_+. \end{aligned} \tag{2}$$

Consider now a deviation by player i to some mixed strategy σ^i that induces uniform marginals. Proposition 1 provides an example of such a mixed strategy. The following analysis is analogous to the proof of Lemma 1. If player $-i$ assigns strictly more than $2m$ units of the resource to battlefield k, then i's payoff from that battlefield is zero. If, however, player $-i$ assigns weakly less than $2m$ to battlefield k, then player i overwhelms her opponent on that battlefield with probability $\frac{2m - s_k^{-i}}{2m+1}$ and achieves a tie with probability $\frac{1}{2m+1}$. Hence, player i's expected payoff from battlefield k is

$$\pi_k^i\left(\sigma^i, s^{-i}\right) = \max\{0, \frac{2m - s_k^{-i}}{2m+1} + \frac{\alpha}{2} \cdot \frac{1}{2m+1}\}.$$

It follows that

$$\begin{aligned} \pi^i\left(\sigma^i, s^{-i}\right) &= \sum_{k=1}^{K} \pi_k^i\left(\sigma^i, s^{-i}\right) \\ &\geq \sum_{k=1}^{K}\left(\frac{2m - s_k^{-i}}{2m+1} + \frac{\alpha}{2} \cdot \frac{1}{2m+1}\right) \\ &= \frac{NK}{2N+K} + \frac{\alpha}{2} \cdot \frac{K^2}{2N+K}. \end{aligned} \tag{3}$$

A straightforward calculation shows that the right-hand side of Eq. (3) strictly exceeds the right-hand side of Eq. (2) if and only if

$$K_+ < K^* \equiv \frac{2}{2-\alpha} \cdot \frac{NK(1-\alpha)}{2N+K}.$$

[11] However, in contrast to Proposition 1, the correlation of the L uniform distributions, one for each pair of battlefields, has been dropped.

[12] As α becomes larger, the conclusion of Proposition 3 weakens. For instance, the conclusion becomes void in the limit case as $\alpha \to 1$.

In particular, in that case, $\pi^i(\sigma^i, s^{-i}) > \pi^i(s^i, s^{-i})$, for any $s^{-i} \in S$. Thus, s^i is never a best response if $K_+ < K^*$. This proves the proposition. $\square$

How strong is the conclusion of Proposition 3 in specific games? For the Arad–Rubinstein game,

$$K^* = \frac{120 \cdot 6}{2 \cdot 120 + 6} = 2.9268.$$

This means that a pure strategy that assigns a positive number of resources to less than three battlefields is never used in equilibrium.

Corollary 2. *In the Arad–Rubinstein game, any pure strategy that allocates the resource to just one or two battlefields is never part of any Nash equilibrium.*

Proof. See the text above. $\square$

Thus, pure strategies such as $(120, 0, 0, 0, 0, 0)'$ and $(80, 40, 0, 0, 0, 0)'$ are never "good" strategies in the example. Regrettably, this leaves a gap when compared to the conclusion of Proposition 2. I.e., we do not know for the Arad–Rubinstein game if strategies such as $(60, 30, 30, 0, 0, 0)'$ that assign strictly positive resources over at least three battlefields and more than 40 soldiers to at least one of the battlefields are "good".[13]

5. Refinements

Given that the Arad–Rubinstein game is a static game, it is natural to check if dominance relationships between strategies might help to narrow down the set of equilibria (e.g., Kohlberg and Mertens (1986)).

Recall that a pure strategy $s^i \in S$ for player $i \in \{A, B\}$ is *weakly dominated* by another pure strategy $\hat{s}^i \in S$ if (i) $\pi^i(s^i, s^{-i}) \leq \pi^i(\hat{s}^i, s^{-i})$ for every pure strategy $s^{-i} \in S$ of the opponent, and (ii) there exists at least one pure strategy $s^{-i} \in S$ for the opponent such that $\pi^i(s^i, s^{-i}) < \pi^i(\hat{s}^i, s^{-i})$. Thus, a pure strategy is weakly dominated by another pure strategy if it never yields a greater payoff than the other strategy, but a strictly lower payoff than the other strategy against at least one pure strategy of the opponent.

It turns out that the elimination of dominated strategies by pure strategies is entirely ineffective for small α.

Proposition 4. *Suppose that $\alpha < \frac{2}{K}$. Then, no pure strategy in $B_\alpha(N, K)$ is weakly dominated by any other pure strategy.*

Proof. By contradiction. Suppose that s is a pure strategy that is weakly dominated by another pure strategy $\hat{s}$. In $B_\alpha(N, K)$, the diagonal entries of the payoff matrix correspond to an outcome with K ties and are, therefore, equal to $K \cdot \frac{\alpha}{2}$. In particular, this is the payoff of $\hat{s}$ against itself. However, as $\hat{s}$ is necessarily different from s, the strategy s bids strictly higher than $\hat{s}$ on at least one battlefield. Therefore, the payoff of s against $\hat{s}$ is at least one. Under the assumption made, this is strictly higher than $K \cdot \frac{\alpha}{2}$. The contradiction shows that no pure strategy can be weakly dominated by any other pure strategy. $\square$

The condition in the proposition is satisfied, in particular, in the Arad–Rubinstein game. There, given that identical choices of pure strategies by the two players lead to ties in all battlefields, $\alpha = 0$ implies that the diagonal entries of the payoff matrix are all zero. In contrast, all of the off-diagonal entries of the payoff matrix are positive because at least one battlefield is won by each player if strategies differ. Therefore, no pure strategy is weakly dominated by any other pure strategy if $\alpha = 0$. This idea of the proof generalizes in a straightforward way to positive but sufficiently small α.[14]

Corollary 3. *In the Arad–Rubinstein game, there are no strategies that are weakly dominated by a pure strategy.*

Proof. In this case, $\alpha = 0$. The claim is, therefore, immediate from Proposition 4. $\square$

6. Further discussion

We first discuss payoff-nonequivalent equilibria (Section 6.1), then the possibility of pure strategy Nash equilibria (Section 6.2), finally the robustness of uniform marginals in the standard model (Section 6.3).

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[13] In the standard model, however, the converse of Proposition 2 holds by Dziubiński (2017, Cor. 2). Thus, an arbitrary pure strategy s^i is used with positive probability in some equilibrium of $B_1(N, K)$ if and only if $s^i_k \leq \frac{2N}{K}$ for every battlefield k. For the reader's convenience, we offer a direct proof. Consider a mixed-strategy equilibrium $\sigma^* = (\sigma^{A,*}, \sigma^{B,*})$ in $B_1(N, K)$. To provoke a contradiction, suppose that there is some player $i \in \{A, B\}$ and some pure strategy s^i in the support of $\sigma^{i,*}$ such that $s^i_k > 2m$ for some battlefield k. As equilibrium strategies in two-person constant-sum game are interchangeable (Osborne and Rubinstein, 1994, p. 23), $\sigma^{i,*}$ is a best response also to the mixed strategy $\sigma^{-i} = \sigma^i$ identified in Proposition 1. Moreover, since the bid vector s^i is chosen with positive probability in the mixed strategy $\sigma^{i,*}$, the pure strategy s^i is likewise a best response to σ^{-i}. But from $s^i_k > 2m$ for some battlefield k, inequality (1) in the proof of Lemma 1 is strict, so that

$$\pi^i(s^i, \sigma^{-i}) < \frac{K \cdot (2N + \alpha K)}{4N + 2K} = \pi^i(\sigma^i, \sigma^{-i}).$$

Thus, s^i is not a best response to σ^{-i} after all. The contradiction proves the assertion.

[14] The parameter constraint on α in Proposition 4 cannot be easily dropped. E.g., in $B_1(120, 6)$, the pure strategy $s^i = (120, 0, 0, 0, 0, 0)'$ is weakly dominated by $\hat{s}^i = (115, 1, 1, 1, 1, 1)'$. For details, see Ewerhart and Kaźmierowski (2024, Ex. 5).

6.1. Payoff-nonequivalent equilibria

In the constant-sum model, all equilibria yield the same payoff. This is not the case with flexible tie-breaking, however. To understand why, let U_{O}^m and U_{E}^m denote uniform marginals on the odd and even numbers between 0 and $2m$, respectively. As shown by Hart (2008, Thm. 7), any pair of strategies that induce marginals in the convex hull of $\{U_{\mathrm{O}}^m, U_{\mathrm{E}}^m\}$ is an equilibrium in $B_1(N,K)$. In particular, any strategy profile $\sigma = (\sigma^i, \sigma^{-i})$, where $\sigma_k^i = U_{\mathrm{O}}^m$ and $\sigma_k^{-i} = U_{\mathrm{E}}^m$ for every battlefield k, is an *asymmetric* equilibrium in $B_1(N,K)$. But any such profile remains an equilibrium for $\alpha \in [0,1]$, because inefficient tie-breaking makes deviations less attractive. Moreover, given the absence of ties, equilibrium payoffs are $K/2$ for both players, which is different from the expression in Proposition 1. Similarly, every mixed strategy σ^i, where $\sigma_k^i = U_{\mathrm{O}}^m$ or $\sigma_k^i = U_{\mathrm{E}}^m$ for every battlefield k is a *symmetric* equilibrium strategy in $B_1(N,K)$. Since deviations make ties less likely, these profiles remain equilibria for $\alpha \in [1,2]$.

6.2. Equilibria in pure strategies

Pure-strategy Nash equilibria, both symmetric and asymmetric, are feasible if parameters are outside of the usual range. For instance, if the payoffs from ties are sufficiently high (i.e., if α is close to or exceeds 2), then the Colonel Blotto game transforms into a coordination game with multiple *symmetric* pure-strategy equilibria.

Proposition 5. *If $\alpha \geq \frac{2\cdot(K-1)}{K}$, then every pure strategy $s^i \in S$ is a symmetric Nash equilibrium strategy in $B_\alpha(N,K)$.*

Proof. Take a symmetric profile $s = (s^i, s^{-i})$ in pure strategies, where $s^i = s^{-i}$. The payoff to both players from s is equal to $K \cdot \frac{\alpha}{2}$. Suppose first that $\alpha \leq 2$. Then, as a deviating player i gives up at least one battlefield, her payoff is bounded from above by $K-1$. But $K-1 \leq K \cdot \frac{\alpha}{2}$ holds by assumption, so that the deviation is not profitable. Suppose next that $\alpha > 2$. Then, changing the matching bid on any battlefield strictly lowers the payoff from that battlefield. Hence, a deviation is not be profitable either. This completes the proof. □

The pathological outcome indicated by Proposition 5 in cases where α is excessively large might provide a rationale for our Assumption 3.

Asymmetric pure-strategy equilibria may emerge when the number of resources N is small relative to the number of battlefields K (and Assumption 3 is in place). Specifically, if $2N \leq K$, adversaries can easily avoid any conflict by dividing the set of battlefields between them.

6.3. The robustness of uniform marginals

This section develops a simple refinement concept for the standard model. For this, let σ_{even} and σ_{odd} denote mixed strategies with marginals uniform on the even and odd integers within $\{0,\ldots,2m\}$, respectively. As shown by Hart (2008) under Assumptions 1 and 2, such strategies exist. Moreover, *any* convex combination of such strategies is an equilibrium strategy in $B_1(N,K)$.

Proposition 6. *The only type of equilibrium in $B_1(N,K)$ that remains an equilibrium for all values of α in a neighborhood of $\alpha = 1$ has uniform marginals.*

Proof. The probability of a tie at any given battlefield is zero in $(\sigma_{\mathrm{even}}, \sigma_{\mathrm{odd}})$, and $(\sigma_{\mathrm{odd}}, \sigma_{\mathrm{even}})$. Similarly, the probability of a tie is $\frac{m+1}{2m+1}$ in $(\sigma_{\mathrm{even}}, \sigma_{\mathrm{even}})$, and $\frac{m}{2m+1}$ in $(\sigma_{\mathrm{odd}}, \sigma_{\mathrm{odd}})$. If player $-i$'s marginal is not uniform and $\alpha < 1$ ($\alpha > 1$), then player i has the incentive to deviate to the parity that is used less (more) often by $-i$. □

The proposition above suggests a simple form of equilibrium selection in finite Colonel Blotto games with standard tie-breaking. Symmetric equilibria $(\sigma_{\mathrm{even}}, \sigma_{\mathrm{even}})$ and $(\sigma_{\mathrm{odd}}, \sigma_{\mathrm{odd}})$ break down for $\alpha < 1$, because players seek to avoid the efficiency loss. Similarly, asymmetric equilibria such as $(\sigma_{\mathrm{even}}, \sigma_{\mathrm{odd}})$ break down for $\alpha > 1$ because players wish to realize the efficiency gain from matching the respective other's strategy. These consideration of robustness might explain why, in the standard model, any equilibrium with uniform marginals, i.e., without parity considerations, is intuitively more appealing than any of the other combinations.

7. Concluding remark

While the equilibrium analysis of discrete Blotto games is of substantial theoretical interest, it falls short in explaining the observations from applied economic research. Consequently, beyond merely characterizing the set of Nash equilibria, it becomes imperative for the theorist to pinpoint "reasonable" choices within the expansive strategy set. We initially speculated that the analysis of adaptive learning might be an avenue to address that issue.[15] However, as kindly pointed out by an anonymous referee, the Blotto game is a static game, and there is clear evidence that players do not think in terms of strategies. Therefore, it seems that further research is needed to resolve this pressing research question.

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[15] Specifically, a simulation of fictitious play (Brown, 1949; Robinson, 1951) resulted in a robust rank correlation of the most likely pure strategies in the learned mixed strategy on the one hand and in the data of Arad and Rubinstein (2012) on the other. We still do not have a good explanation for that observation.

Declaration of competing interest

None.

Data availability

No data was used for the research described in the article.

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